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Medium Effects on the CP phases and Dynamical Mixing Angles in the Neutrino Mixing Matrix

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Abstract

The concepts of effective and dynamical neutrino mixing matrices are introduced in order to describe the behavior of neutrinos in matter. The former relates weak eigenstates to mass eigenstates, whereas the latter relates weak eigenstates to energy eigenstates in matter. It is shown that the dynamical mixing angles enable us to express the neutrino survival probability in the Sun without any resort to the Landau-Zener transition probability for the non-adiabatic process. Also discussed are effective CP violating phases that appear in the effective and dynamical mixing matrices in matter. Both two and three generation cases are discussed using the solar neutrinos as an

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I. INTRODUCTION

The Mikheyev-Smirnov-Wolfenstein (MSW) [1,2] effect has been considered for some time as the most plausible mechanism to explain the solar neutrino puzzles [3–6]. This effect dramatically enhances the conversion of a flavor neutrino (ν_e in the case of solar neutrinos) into another when certain conditions among the intrinsic neutrino properties and the nature of matter are met in a medium. When the matter density in the Sun changes slowly enough (to be determined by the intrinsic properties such as masses and mixing angles and by the neutrino energy) so that the neutrino conversion from one flavor to another takes place adiabatically, the MSW effects are known to be well described by the effective masses and mixing angles in matter. These effective values at any given time or distance from the production in matter are commonly derived from the equation of motion for the weak eigenstates in matter by a simple matrix diagonalization under the assumption that the matter density is uniform.

However, when the time (or distance) variation of the density in matter is sufficiently rapid, transitions among the effective mass eigenstates take place and the process becomes non-adiabatic. In this case, the role of the effective values becomes vague. In order to overcome this, the neutrino survival probability is approximately expressed in terms of the Landau-Zener transition probability [7–14]. The other alternative is, of course, to solve numerically the equation of motion for the weak eigenstates in matter. It has been recognized, however, that to do so is prohibitively difficult when all three generations of the neutrinos are involved.

In this paper,we first introduce two concepts of the mixing angles in matter, effective and dynamical mixing angles and the corresponding mixing matrices. It will be shown that the dynamical mixing angles enable us to calculate the survival probability without resort to the Landau-Zener transition probability. The effective mixing angles defined here are equivalent to those that have been commonly used in the past, i.e., they appear in the effective mixing matrix which relates the weak eigenstates and the mass eigenstates in matter. As mentioned already, these angles are sufficient to describe the adiabatic process and have also been used for the non-adiabatic case together with the Landau-Zener transition probability.

The dynamical mixing angles are defined to appear in the mixing matrix which relates the weak eigenstates and the *energy* eigenstates in matter. (Note that even in the case when there appear transitions among the mass eigenstates, the energy eigenstates are well defined.) It will be shown that the neutrino survival probability can be expressed in terms of the dynamical mixing angle alone even in the case of non-adiabatic processes. Since in

this approach no approximations are involved in calculating the survival probability, the results are exact whereas the usual treatment with the use of the Landau-Zener transition probability relies on some approximations.

The second subject to be discussed in this paper is the medium effect on the CP violating phases already present in the neutrino mixing matrix. In the case of general mixings, it has been known that the CP phases would not significantly modify the neutrino transition probabilities, in particular when the oscillating terms are averaged out. Therefore, an observation of the CP violating effects is rather difficult, if not impossible. This general conclusion, however, needs not to be correct in the case of maximal mixings. The possible solution of the atmospheric neutrinos and the so-called large angle solution of the solar neutrino problem require rather large mixing angles, suggesting the possibility of maximal mixings of three neutrinos [15,16]. In this case, it is necessary to have a CP violating phase. That is, maximal mixings are impossible without CP violating phases [17].

In this paper, we investigate how the medium effects would modify the CP violating phases in the mixing matrix in the vacuum. It will be shown that in the case of three generations of neutrinos, in addition to numerical changes in the original vacuum CP phase, the medium effects induce additional *effective* CP phases which, of course, would vanish when the matter effects are turned off. Although it is necessary to have additional CP phases in matter, they are not independent of the original vacuum CP phases. In the two generation case, no additional CP phase appears in matter.

This paper has been organized as follows: In Section II we introduce two mixing matrices, the effective mixing matrix and the dynamical mixing matrix. Section III is devoted to a detail discussion of the effective mixing matrices in two and three generations of neutrinos. In particular, we demonstrate how additional induced phases appear in a medium in the case of three generations. In Section IV, we discuss the dynamical mixing matrices. After presenting a simple example of the two generation case, we discuss the case of three generations with non-zero CP. We summarize our results and conclude in Section V.

II. DEFINITIONS OF EFFECTIVE AND DYNAMICAL MIXING MATRICES

Let us begin with the wave equation for neutrinos propagating through a medium with a time-dependent potential, $V(t)$, being felt by the neutrinos *a la* MSW effects

$$i\frac{d}{dt}|\Psi_w(t)>=\frac{1}{2E}M^w|\Psi_w(t)>, \tag{1}$$

where $|\Psi_w(t)>$ is the neutrino wave function in the weak (flavor) basis and M^w is the time evolution matrix in the weak basis which dictates the motion of the weak eigenstate neutrinos. First, we define the *effective mixing matrix* U_{eff} which satisfies the following equation

$$U_{eff}^{\dagger} M^w U_{eff} = M^m, \tag{2}$$

where M^m is the diagonal mass matrix in the mass basis. Solving the neutrino wave equation Eq. (1) is equivalent to finding the effective mixing matrix U_{eff} in Eq. (2) in the case where neutrinos propagate in a medium with a uniform density ($V(t) =$ constant). The result is sufficient to describe the adiabatic MSW process in a medium.

Next, we introduce a unitary matrix U which diagonalizes the time-evolution matrix for a time-dependent $V(t)$. By transforming the wave function in the weak basis with the unitary matrix U, we obtain the wave function in the energy basis such that its time evolution is of the form

$$|\Psi_e(t)> = e^{-i\frac{1}{2E}\int_0^t M^e(t)dt}|\Psi_e(0)>, \tag{3}$$

where $M^e(t)$ is diagonal and the subscript e in the wave function Ψ_e denotes the energy eigenstate. In other words, the above-introduced unitary matrix U relates $|\Psi_w(t)>$ and $|\Psi_e(0)>$ for a time-dependent $V(t)$ as

$$|\Psi_w(t)> = U(t)|\Psi_e(t)> . \tag{4}$$

By inserting Eq. (4) into Eq. (1), we can obtain a differential equation for the matrix U. From now on , we will call this U the *dynamical mixing matrix* and use the notation U_{dyn}. The U_{dyn} satisfies the differential equation

$$U_{dyn}^{\dagger} M^w U_{dyn} = M^e + 2EiU_{dyn}^{\dagger}\dot{U}_{dyn}, \tag{5}$$

where M^e is a diagonal matrix whose elements are composed of time varying energies of the Hamiltonian $M^w/2E$.

The form of the dynamical mixing matrix U_{dyn} does not depend on whether the neutrinos are Dirac or Majorana type. For the two generation case, the dynamical mixing matrix is a 2×2 matrix with one dynamical mixing angle and one dynamical phase if the effective potential $V(t)$ is not a constant in time. For the three generation case, the dynamical mixing matrix is a 3×3 matrix with three dynamical mixing angles and three dynamical phases.

Summarizing the above, the effective mixing matrix relates the weak eigenstates and the mass eigenstates

$$| \text{ weak eigenstates} >= U_{eff} | \text{mass eigenstates} >, \tag{6}$$

whereas the dynamical mixing matrix relates the weak eigenstates and the energy eigenstates

$$| \text{ weak eigenstates} >= U_{dyn} | \text{energy eigenstates} > . \tag{7}$$

Usually, it is easy to calculate the effective mixing matrix by using simple algebra , i.e., it is an eigenvalue and eigenvector problem. The eigenvalues correspond to the effective masses and an appropriate assemble of eigenvectors is nothing but the effective mixing matrix defined by Eq. (2). As for the dynamical mixing matrix, however, it is non-trivial to obtain the dynamical mixing angles, phases and the time-dependent energy eigenvalues because it is equivalent to solving the Schrödinger equation of the neutrino wave function given in Eq. (1). Therefore, one often resorts to numerical methods to solve Eq.(1).

III. EFFECTIVE NEUTRINO MIXING MATRICES

A. Two Generation Case

As a simple example, let us consider the two generation mixing of Majorana neutrinos to investigate how the CP phase in vacuum would be modified in matter. (In the case of Dirac neutrinos, the phase is trivial, that is, zero in matter as well as in vacuum.) In general, the mixing matrix in vacuum can be written as

$$U_0 = \begin{pmatrix} \cos\theta & \sin\theta e^{i\omega} \\ -\sin\theta e^{-i\omega} & \cos\theta \end{pmatrix} . \tag{8}$$

In Eq.(8) and hereafter, a quantity with subscript 0 always denotes its value in vacuum. The mass matrix in the weak basis can be obtained from the mass matrix in the mass basis as

$$M_0^w = U_0 M_0^m U_0^\dagger \equiv \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} , \tag{9}$$

where $M_0^m = \mathrm{diag}(m_1^2, m_2^2)$.

In matter, a quantity $A(t) \equiv 2E_\nu V(t)$, which is the amount of the increase in the effective mass squared in matter for ν_e, is inserted in the 1-1 element of the mass matrix

$$M^w = \begin{pmatrix} m_{11} + A & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{10}$$

with $A = 2E_\nu V = 2\sqrt{2}G_F E_\nu N_e$. (We ignore the contributions from neutral current interactions.) The G_F, E_ν, and N_e represent, respectively, the Fermi coupling constant, the energy of the neutrino under consideration, and the electron number density in a medium. Let us denote the effective mixing matrix in matter as

$$U_{eff} = \begin{pmatrix} \cos\bar{\theta} & \sin\bar{\theta}e^{i\bar{\omega}} \\ -\sin\bar{\theta}e^{-i\bar{\omega}} & \cos\bar{\theta} \end{pmatrix}. \tag{11}$$

This effective mixing matrix U_{eff} diagonalizes the effective mass matrix in the weak basis,i.e., $U^\dagger_{eff}M^w U_{eff} = M^m$, where $M^m = \mathrm{diag}(\mu_1^2, \mu_2^2)$. The effective masses μ_1, μ_2 and the effective mixing angle $\bar{\theta}$ are well known and are given by (For recent reviews, see for example, [11,18])

$$\mu_1^2 = \frac{m_1^2 + m_2^2}{2} + \frac{A}{2} - \frac{1}{2}\sqrt{(A - \Delta_0^2\cos 2\theta)^2 + \Delta_0^4 \sin^2 2\theta}, \tag{12}$$

$$\mu_2^2 = \frac{m_1^2 + m_2^2}{2} + \frac{A}{2} + \frac{1}{2}\sqrt{(A - \Delta_0^2\cos 2\theta)^2 + \Delta_0^4 \sin^2 2\theta}, \tag{13}$$

$$\tan 2\bar{\theta} = \frac{\tan 2\theta}{1 - A/\Delta_0^2 \cos 2\theta}, \tag{14}$$

where $\Delta_0^2 \equiv m_2^2 - m_1^2$. In the two generation case, a simple algebra shows that the effective phase $\bar{\omega}$ remains unaffected by matter, i.e., $\bar{\omega} = \omega$. However, this is not the case for the three generation case as well as for the two generation case of the dynamical mixixing matrix, as we will see later. The above information is sufficient to describe the adiabatic conversion processes of ν_e in the Sun.

B. Three Generation Case

Now let us consider the case of three generations of neutrinos by generalizing the case discussed above. The representation of the unitary mixing matrix [19–21] depends on the neutrino type. Majorana neutrinos can have three non-zero CP phases in vacuum whereas Dirac neutrinos can have only one CP phase. For simplicity, we will consider the case where only one non-zero CP phase appears in vacuum. The result which we will present below can easily be generalized to the case with more than one vacuum CP phases.

We have found it convenient to describe the neutrino mixing with the use of the modified Maiani representation as advocated in Particle Data Group [22] given by

$$U_0 = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix}, \tag{15}$$

where $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$. (The mixing angles θ_{ij} and phase δ in Eq. (15) are all vacuum values although they do not have the subscript 0.)

The mass matrix in the weak basis is related to the mass matrix in the mass basis as

$$M_0^w = U_0 M_0^m U_0^\dagger \equiv \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \tag{16}$$

where $M_0^m = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$. In a way analogous to the two generation case, the matter effect is inserted in the mass matrix:

$$M^w = \begin{pmatrix} m_{11}+A & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}. \tag{17}$$

Now, solving the problem in matter amounts to diagonalizing the Hamiltonian or solving the time evolution matrix in Eq. (1). In the adiabatic case or in the case of a uniform matter density, it is equivalent to finding the effective mass matrix M^m and the effective mixing matrix U_{eff} from the equation $U_{eff}^\dagger M^w U_{eff} = M^m$, where $M^m = \mathrm{diag}(\mu_1^2, \mu_2^2, \mu_3^2)$. The eigenvalues are

$$\mu_1^2 = 2\alpha^{1/3}\cos(\frac{\beta+2\pi}{3}) - \frac{b}{3}, \tag{18}$$

$$\mu_2^2 = 2\alpha^{1/3}\cos(\frac{\beta-2\pi}{3}) - \frac{b}{3}, \tag{19}$$

$$\mu_3^2 = 2\alpha^{1/3}\cos(\frac{\beta}{3}) - \frac{b}{3}, \tag{20}$$

where

$$\alpha \equiv \sqrt{q^2 + |p^3+q^2|}, \qquad \beta \equiv \arctan(\frac{\sqrt{|p^3+q^2|}}{q}),$$

and

$$p = \frac{c}{3} - \frac{b^2}{9}, \qquad q = -\frac{d}{2} + \frac{bc}{6} - \frac{b^3}{27},$$

with $b = -\mathrm{Tr}M^w$, $c = \frac{1}{2}[(\mathrm{Tr}M^w)^2 - \mathrm{Tr}(M^w)^2]$, and $d = -\mathrm{det}M^w$.

Let us denote the eigenvector corresponding to the third eigenvalue as a column vector $(c_1, c_2, c_3)^T$, where T denotes transpose. Solving the equations for c_1, c_2 and c_3, we obtain the ratios

$$\frac{c_2}{c_1} = \frac{(m_{33}-\mu_3^2)m_{21} - m_{23}m_{31}}{m_{23}m_{32} - (m_{22}-\mu_3^2)(m_{33}-\mu_3^2)}, \tag{21}$$

$$\frac{c_3}{c_1} = \frac{(m_{22} - \mu_3^2)m_{31} - m_{32}m_{21}}{m_{23}m_{32} - (m_{22} - \mu_3^2)(m_{33} - \mu_3^2)}. \tag{22}$$

In matter, the phases of c_2/c_1 and c_3/c_1 are no longer the same due to the presence of the non-zero vacuum CP phase δ. This is the reason why we are forced to introduce two additional phases for the mixing matrix to be unitary and self-consistent. Thus, we can not use the same form as the modified Maiani representation given in Eq. (15) in order to describe the new effective matrix in matter.

Let us take a representation in matter which has two additional *induced* phases, ξ and η which appear with θ_{23} and θ_{12}, respectively,

$$U_{eff} = \begin{pmatrix} \bar{c}_{12}\bar{c}_{13} & \bar{s}_{12}\bar{c}_{13}e^{i\eta} & \bar{s}_{13}e^{-i\bar{\delta}} \\ -\bar{s}_{12}\bar{c}_{23}e^{-i\eta} - \bar{c}_{12}\bar{s}_{13}\bar{s}_{23}e^{i(\bar{\delta}-\xi)} & \bar{c}_{12}\bar{c}_{23} - \bar{s}_{12}\bar{s}_{13}\bar{s}_{23}e^{i(\bar{\delta}+\eta-\xi)} & \bar{c}_{13}\bar{s}_{23}e^{-i\xi} \\ \bar{s}_{12}\bar{s}_{23}e^{-i(\eta-\xi)} - \bar{c}_{12}\bar{s}_{13}\bar{c}_{23}e^{i\bar{\delta}} & -\bar{c}_{12}\bar{s}_{23}e^{i\xi} - \bar{s}_{12}\bar{s}_{13}\bar{c}_{23}e^{i(\bar{\delta}+\eta)} & \bar{c}_{13}\bar{c}_{23} \end{pmatrix}. \tag{23}$$

where $\bar{s}_{ij} \equiv \sin\bar{\theta}_{ij}$, $\bar{c}_{ij} \equiv \cos\bar{\theta}_{ij}$. A barred quantity denotes its effective value in matter. The two additional phases, ξ and η, are *induced* in matter due to the presence of the original non-zero CP phase in vacuum. With the representation Eq. (23), we can obtain all the effective values in matter.

The two effective mixing phases are given by

$$\tan\bar{\delta} = \frac{\mathrm{Im}(c_3/c_1)}{\mathrm{Re}(c_3/c_1)}, \tag{24}$$

$$\tan(\bar{\delta} - \xi) = \frac{\mathrm{Im}(c_2/c_1)}{\mathrm{Re}(c_2/c_1)}, \tag{25}$$

and the effective mixing angles are

$$\tan\bar{\theta}_{13} = \frac{1}{\sqrt{|c_2/c_1|^2 + |c_3/c_1|^2}}, \tag{26}$$

$$\cos\bar{\theta}_{23} = \frac{|c_3/c_1|}{\sqrt{|c_2/c_1|^2 + |c_3/c_1|^2}}. \tag{27}$$

In order to obtain the remaining mixing angle $\bar{\theta}_{12}$ and the phase η, we must solve another eigenvector for an eigenvalue, for example, μ_1^2. Let us denote the eigenvector as a column vector $(d_1, d_2, d_3)^T$. Solving the equations for d_1, d_2 and d_3, we obtain the ratio of d_2 to d_1

$$\frac{d_2}{d_1} = \frac{(m_{33} - \mu_1^2)m_{21} - m_{23}m_{31}}{m_{23}m_{32} - (m_{22} - \mu_1^2)(m_{33} - \mu_1^2)}. \tag{28}$$

After comparing with the ratios of the 1-1 and 1-2 mixing matrix elements, we obtain the effective phase η and the effective mixing angle $\bar{\theta}_{12}$ as

$$\tan\eta = -\frac{\tan\bar{\theta}_{13}\sin\bar{\theta}_{23}\sin(\bar{\delta}-\xi)+\mathrm{Im}(d_2/d_1)}{\tan\bar{\theta}_{13}\sin\bar{\theta}_{23}\cos(\bar{\delta}-\xi)+\mathrm{Re}(d_2/d_1)}, \tag{29}$$

$$\tan\bar{\theta}_{12} = -(\tan\bar{\theta}_{13}\sin\bar{\theta}_{23}\cos(\bar{\delta}-\xi)+\mathrm{Re}(d_2/d_1))\frac{\cos\bar{\theta}_{13}}{\cos\bar{\theta}_{23}\cos\eta}. \tag{30}$$

From Eqs. (24) and (29), we can see that the two additional induced phases ξ and η are generated by the non-zero vacuum CP phase δ. That is, if we set $\delta = 0$, then Eq. (24) shows that the effective phase $\bar{\delta}$ also vanishes because c_3/c_1 cannot have an imaginary part. Similarly, it is easy to see that the two additional phases ξ and η vanish if $\delta = 0$. We emphasize that the two phases ξ and η are not new and independent phases, but instead they are related to the non-zero vacuum CP phase δ.

In order to exhibit this property quantitatively, we use the solar neutrinos as an example. Most solar neutrinos are produced in the region $x \leq 0.25$ where x is the ratio of the distance R from the center of the Sun and the radius of the Sun $R_\odot$. In the Sun, the electron number density distribution [23,24] is well fitted by $\log_{10}(N_e(x)/N_A) = 2.32-4.17x-0.000125/[x^2+0.5^2]$ where N_A is Avogadro's number per cm^3. The quantity A in the center of the Sun is $A \cong 3.17\times10^{-5}(E_\nu/\mathrm{MeV})\mathrm{eV}^2$. With the solar neutrino energy $E_\nu \leq 14\mathrm{MeV}$, the largest value of A in the Sun is of the order of $10^{-3}\mathrm{eV}^2$. Therefore, if we assume small vacuum mixing angles, there are two resonance points ($\bar{\theta}_{ij} = \pi/4$) inside the Sun when the mass squared difference $\Delta_{31} = m_3^2 - m_1^2$ is less than $10^{-3}\mathrm{eV}^2$.

Fig. 1 and Fig. 2 show the typical behavior of the three effective mixing angles and three phases in the case where the neutrino mass squared differences are taken as $\Delta_{21} \sim 10^{-4}\mathrm{eV}^2$, $\Delta_{31} \sim 10^{-2}\mathrm{eV}^2$. In Fig. 1, we have taken the vacuum mixing angles $\theta_{12} = \theta_{13} = \theta_{23} = 0.05$ and the vacuum CP phase $\delta = 0.5$. Therefore, we can see that, in contrast to the case of the effective mixing angles and effective masses which are dramatically affected by the medium, the CP phase $\bar{\delta}$ remains unchanged, and ξ and η are induced for large values of A in the adiabatic processes. Fig. 2 is for the vacuum mixing angles $\theta_{12} = \theta_{23} = \pi/4$, $\theta_{13} = 0.6155$, and the vacuum CP phase $\delta = \pi/2$. This is the maximal mixing case. In Fig. 2, the effective mixing angles $\bar{\theta}_{12}$ and $\bar{\theta}_{13}$ change from $\theta_{12} = \pi/4$ and $\theta_{13} = 0.6155$ in vacuum to $\bar{\theta}_{12} \sim \bar{\theta}_{13} \sim \pi/2$ in the solar interior, whereas the mixing angle $\bar{\theta}_{23}$ and the phase $\bar{\delta}$ also practically remain unchanged. Again, the induced ξ and η appear when A becomes large. This confirms the assertion made in [25,26] that the matter effects are practically absent in the maximal mixing case.

In general, the non-zero CP phase contributes to the transition probability among the flavor neutrino states. For simplicity, we consider the averaged survival probability of ν_e in the adiabatic approximation. The survival probability is given by

$$
\begin{aligned}
< P(\nu_e \to \nu_e; \text{adiabatic}) > \; = & \\
& (\bar{c}_{12}\bar{c}_{13}c_{12}c_{13})^2 + (\bar{s}_{12}\bar{c}_{13}s_{12}c_{13})^2 + (\bar{s}_{13}s_{13})^2 \\
& +2\bar{c}_{12}\bar{c}_{13}\bar{s}_{12}\bar{c}_{13}c_{12}c_{13}s_{12}c_{13}\cos\eta \\
& +2\bar{c}_{12}\bar{c}_{13}\bar{s}_{13}c_{12}c_{13}s_{13}\cos(\bar{\delta}-\delta) \\
& +2\bar{s}_{12}\bar{c}_{13}\bar{s}_{13}s_{12}c_{13}s_{13}\cos(\eta+\bar{\delta}-\delta). \qquad (31)
\end{aligned}
$$

Although three phases are present in Eq. (31), since they appear as $\cos\eta, \cos(\bar{\delta}-\eta)$, and $\cos(\eta+\bar{\delta}-\delta)$, all of which are close to unity, the CP phases are almost impossible to detect in practice. (Note in the figure that $\bar{\delta} \sim \delta$ and $\xi, \eta \sim 0$).

IV. DYNAMICAL NEUTRINO MIXING MATRICES

A. Two Generation Case

Now let us study the dynamical mixing matrix in detail. We will first consider the two generation case. From the definition of the dynamical mixing matrix given by Eq. (5), we can write the differential matrix equation in a 2×2 matrix form. As it becomes clear later, at least one non-zero phase is necessary independently of the neutrino type even in the two generation case. With the following notations

$$
U_{dyn} \equiv \begin{pmatrix} \cos\psi & \sin\psi e^{i\alpha} \\ -\sin\psi e^{-i\alpha} & \cos\psi \end{pmatrix}, \qquad (32)
$$

$$
M^e \equiv \begin{pmatrix} \mathcal{E}_1 & 0 \\ 0 & \mathcal{E}_2 \end{pmatrix}, \qquad (33)
$$

and

$$
M^w = \begin{pmatrix} -\frac{\Delta_0^2}{2}\cos 2\theta + A(t) & \frac{\Delta_0^2}{2}\sin 2\theta \\ \frac{\Delta_0^2}{2}\sin 2\theta & \frac{\Delta_0^2}{2}\cos 2\theta \end{pmatrix}, \qquad (34)
$$

Eq. (5) leads to the two coupled equations for ψ and α

$$
\dot{\psi} = -\frac{\Delta_0^2}{4E}\sin 2\theta \sin\alpha, \qquad (35)
$$

$$
\dot{\alpha} = -\frac{\Delta_0^2}{2E}\sin 2\theta\cos\alpha\cot 2\psi + \frac{\Delta_0^2\cos 2\theta - A(t)}{2E}. \qquad (36)
$$

Here, it is easy to see that if there is no phase (i.e., $\alpha = 0$), it is impossible to describe the case where A changes in time since with $\alpha = 0$ Eq. (36) is valid only at a fixed value of

t. In the above, the mass matrix in the weak basis in Eq. (34) has been obtained from the mass matrix Eq. (10) by subtracting $(m_1^2 + m_2^2)/2$ from the diagonal elements in Eq. (10). In addition, we obtain the energy eigenvalues, $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$,

$$\mathcal{E}_1(t) = -\frac{\Delta_0^2}{2}\cos 2\theta + A(t) - \frac{\Delta_0^2}{2}\sin 2\theta \cos\alpha \tan\psi, \tag{37}$$

$$\mathcal{E}_2(t) = \frac{\Delta_0^2}{2}\cos 2\theta - \frac{\Delta_0^2}{2}\sin 2\theta \cos\alpha \tan\psi. \tag{38}$$

Equations (35) and (36) are coupled non-linear differential equations. Similar equations have previously been obtained in [27] as a means to describe the non-adiabatic process without the use of the Landau-Zener transition probability. The initial conditions for ψ and α are fixed by the vacuum values. Given $A(t)$, we can solve the equations numerically. For $E/(m_2^2 - m_1^2) \to \infty$, $\psi(t)$ and $\alpha(t)$ approach the vacuum values, which can be seen from the usual treatment with the Landau-Zener formula. Fig. 3 (a) (Fig. 3 (b)) shows the dynamical mixing angle and phase in the case of the neutrino energy $E_\nu = 10$ MeV, the vacuum mixing angle $\theta = 0.1$, and $\Delta_0^2 = 10^{-4}\mathrm{eV}^2$ ($\Delta_0^2 = 10^{-6}\mathrm{eV}^2$). The difference between the effective and dynamical mixing angles depends sensitively on Δ_0^2. Fig. 3 (c) shows the dynamical mixing angle and phase in the maximal mixing case of two neutrinos with the neutrino energy $E_\nu = 10$ MeV and the mass squared difference $\Delta_0^2 = 10^{-6}\mathrm{eV}^2$. The abscissa x indicates the ratio of the distance of interest from the surface of the Sun to the solar radius, i.e., $x = 0$ corresponds to the surface of the Sun and $x = 1$ the center of the Sun. The sign of the dynamical phase α depends on the direction of the neutrino motion, i.e., the sign of α is reversed as ν_e travels inwards from the solar surface. The behavior of α as a function of x is quite remarkable as $x \to 1$.

The time averaged survival probability for ν_e is given by

$$< P(\nu_e \to \nu_e) >= \cos^2\psi \cos^2\theta + \sin^2\psi \sin^2\theta. \tag{39}$$

We emphasize here that although the expression in Eq. (39) is identical to the usual survival probability for the adiabatic process when ψ is replaced by the effective mixing angle $\bar{\theta}$, it is an exact one which can describe the non-adiabatic process without using the Landau-Zener transition probability. In Eq. (39) the dynamical phase α has disappeared in the process of time average. Fig. 4 shows the survival probabilities given by Eq. (39) as functions of E_ν/Δ_0^2 for three examples with the vacuum mixing angles $\theta = 0.01$, $\theta = 0.1$ and $\theta = \pi/4$ (maximal mixing in two generations). The transition probability for the maximal mixing case is practically constant at 0.5, in the region of $10^{-10} \le E_\nu/\Delta_0^2 \le 10^{-15}$ and the neutrino oscillation in matter cannot be differentiated from the pure vacuum oscillation. However it

does not mean that the dynamical mixing angle ψ remains constant in matter. The value of the angle ψ approaches to $\pi/2$ near the center of the Sun. In Fig. 4 the survival probabilities calculated using the Landau-Zener transition probability (dotted lines for $\theta = 0.01, \theta = 0.1$) are compared with the exact results based on the use of dynamical mixing angle.

B. Three Generation Case

In the case of three generations, there are six coupled differential equations and three relations to be solved. The unknown variables are three dynamical mixing angles, three dynamical phases and three energy eigenvalues. Let us write the dynamical neutrino mixing matrix in the three generation as

$$U_{dyn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{-i\gamma} \\ 0 & -s_{23}e^{i\gamma} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\beta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\beta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12}e^{i\alpha} & 0 \\ -s_{12}e^{-i\alpha} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{40}$$

where $c_{ij} \equiv \cos\psi_{ij}, s_{ij} \equiv \sin\psi_{ij}$ with α, β and γ denoting the dynamical CP phases in matter. From the off-diagonal elements of Eq. (5) we can obtain the following differential equations for three mixing angles and three mixing phases,

$$\begin{pmatrix} \dot{\psi}_{12} \\ \dot{\psi}_{13} \\ \dot{\psi}_{23} \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \frac{1}{2E}\mathcal{B}^{-1} \begin{pmatrix} \mathrm{Re}(U^\dagger_{dyn}M^wU_{dyn})_{12} \\ \mathrm{Im}(U^\dagger_{dyn}M^wU_{dyn})_{12} \\ \mathrm{Re}(U^\dagger_{dyn}M^wU_{dyn})_{13} \\ \mathrm{Im}(U^\dagger_{dyn}M^wU_{dyn})_{13} \\ \mathrm{Re}(U^\dagger_{dyn}M^wU_{dyn})_{23} \\ \mathrm{Im}(U^\dagger_{dyn}M^wU_{dyn})_{23} \end{pmatrix}. \tag{41}$$

Here the 6×6 matrix $\mathcal{B}$ is given by

$$\mathcal{B} = \begin{pmatrix} -s_\alpha & 0 & c_{12}^2 s_{13} s_{\beta-\gamma} - s_{12}^2 s_{13} s_{2\alpha+\beta-\gamma} & -s_{12}c_{12}c_\alpha & -s_{12}c_{12}s_{13}^2 c_\alpha & b_{16} \\ c_\alpha & 0 & c_{12}^2 s_{13} c_{\beta-\gamma} + s_{12}^2 s_{13} c_{2\alpha+\beta-\gamma} & -s_{12}c_{12}s_\alpha & -s_{12}c_{12}s_{13}^2 s_\alpha & b_{26} \\ 0 & c_{12}s_\beta & s_{12}c_{13}s_{\alpha-\gamma} & 0 & c_{12}s_{13}c_{13}c_\beta & b_{36} \\ 0 & c_{12}c_\beta & -s_{12}c_{13}c_{\alpha-\gamma} & 0 & -c_{12}s_{13}c_{13}s_\beta & b_{46} \\ c_{12}c_{13}s_\gamma & s_{12}s_{\alpha+\beta} & 0 & 0 & s_{12}s_{13}c_{13}c_{\alpha+\beta} & b_{56} \\ c_{12}c_{13}c_\gamma & s_{12}c_{\alpha+\beta} & 0 & 0 & -s_{12}s_{13}c_{13}s_{\alpha+\beta} & b_{66} \end{pmatrix}, \tag{42}$$

where

$$b_{16} = s_{12}c_{12}(1+s_{13}^2)s_{23}^2 c_\alpha + s_{12}^2 s_{13}s_{23}c_{23}c_{2\alpha+\beta-\gamma} - c_{12}^2 s_{13}s_{23}c_{23}c_{\beta-\gamma},$$
$$b_{26} = s_{12}c_{12}(1+s_{13}^2)s_{23}^2 s_\alpha + s_{12}^2 s_{13}s_{23}c_{23}s_{2\alpha+\beta-\gamma} + c_{12}^2 s_{13}s_{23}c_{23}s_{\beta-\gamma},$$
$$b_{36} = -s_{12}c_{13}s_{23}c_{23}c_{\alpha-\gamma} - c_{12}s_{13}c_{13}s_{23}^2 c_\beta,$$
$$b_{46} = -s_{12}c_{13}s_{23}c_{23}s_{\alpha-\gamma} + c_{12}s_{13}c_{13}s_{23}^2 s_\beta,$$
$$b_{56} = -s_{12}s_{13}c_{13}s_{23}^2 c_{\alpha+\beta} + c_{12}c_{13}s_{23}c_{23}c_\gamma,$$
$$b_{66} = s_{12}s_{13}c_{13}s_{23}^2 s_{\alpha+\beta} - c_{12}c_{13}s_{23}c_{23}s_\gamma.$$

The energy eigenvalues are

$$\begin{aligned}\mathcal{E}_1(t) &= (U_{dyn}^\dagger M^w U_{dyn})_{11} - 2E\{\dot\alpha s_{12}^2 - \dot\beta c_{12}^2 s_{13}^2 - \\ &\quad \dot\gamma[(s_{12}^2 - c_{12}^2 s_{13}^2)s_{23}^2 - 2s_{12}c_{12}s_{13}s_{23}c_{23}c_{\alpha+\beta-\gamma}] - 2\dot\psi_{23}s_{12}c_{12}s_{13}s_{\alpha+\beta-\gamma}\},\end{aligned} \tag{43}$$

$$\begin{aligned}\mathcal{E}_2(t) &= (U_{dyn}^\dagger M^w U_{dyn})_{22} + 2E\{\dot\alpha s_{12}^2 + \dot\beta s_{12}^2 s_{13}^2 + \\ &\quad \dot\gamma[(c_{12}^2 - s_{12}^2 s_{13}^2)s_{23}^2 - s_{12}c_{12}s_{13}s_{23}c_{23}c_{\alpha+\beta-\gamma}] - \dot\psi_{23}s_{12}c_{12}s_{13}s_{\alpha+\beta-\gamma}\},\end{aligned} \tag{44}$$

$$\mathcal{E}_3(t) = (U_{dyn}^\dagger M^w U_{dyn})_{33} - 2E\{\dot\beta s_{13}^2 + \dot\gamma c_{13}^2 s_{23}^2\}. \tag{45}$$

The averaged transition probability is of the following form:

$$< P(\nu_e \to \nu_e) >= \sum_{i=1}^{3} |(U_{dyn}^*)_{i1}(U_0)_{i1}|^2. \tag{46}$$

In the three generation case, the dynamical phases contribute through $(U_{dyn}^*)_{i1}$ to the transition probability in contrast to the two generation case. The formulas in the three generation case are reduced to the corresponding expressions in the two generation case, when we take $\theta_{13} = \theta_{23} = 0$ and $m_{3i} = m_{i3} = 0 (i = 1, 2, 3)$. In the case of three generation of neutrinos, however, it is prohibitively difficult to solve the coupled differential equation (41)- (45) numerically and such a study is beyond the scope of the present work.

V. CONCLUSION

In this paper we have introduced the effective mixing matrix that relates weak eigenstates to mass eigenstates, and the dynamical mixing matrix that connects weak eigenstates and energy eigenstates. Using these definitions, we have studied the effective mixing matrices for the two and three flavor mixing cases, with emphasis on the nature and behavior of the CP phases in matter. It has been shown that in the case of three generation mixing, two additional induced phases appear when a vacuum CP phase is non-zero. Analysis of the dynamical mixing matrices for the two and three flavor mixing cases has also been presented.

In contrast to the effective mixing matrix of the two flavor mixing, in the dynamical mixing matrix, it is necessary to introduce one additional phase in a medium in order to be self consistent. This does not depend on the neutrino type. We have used the solar neutrinos to demonstrate the differences involved in the behavior of the CP phases in matter, and the differences between the effective and dynamical mixing matrices.

ACKNOWLEGMENTS

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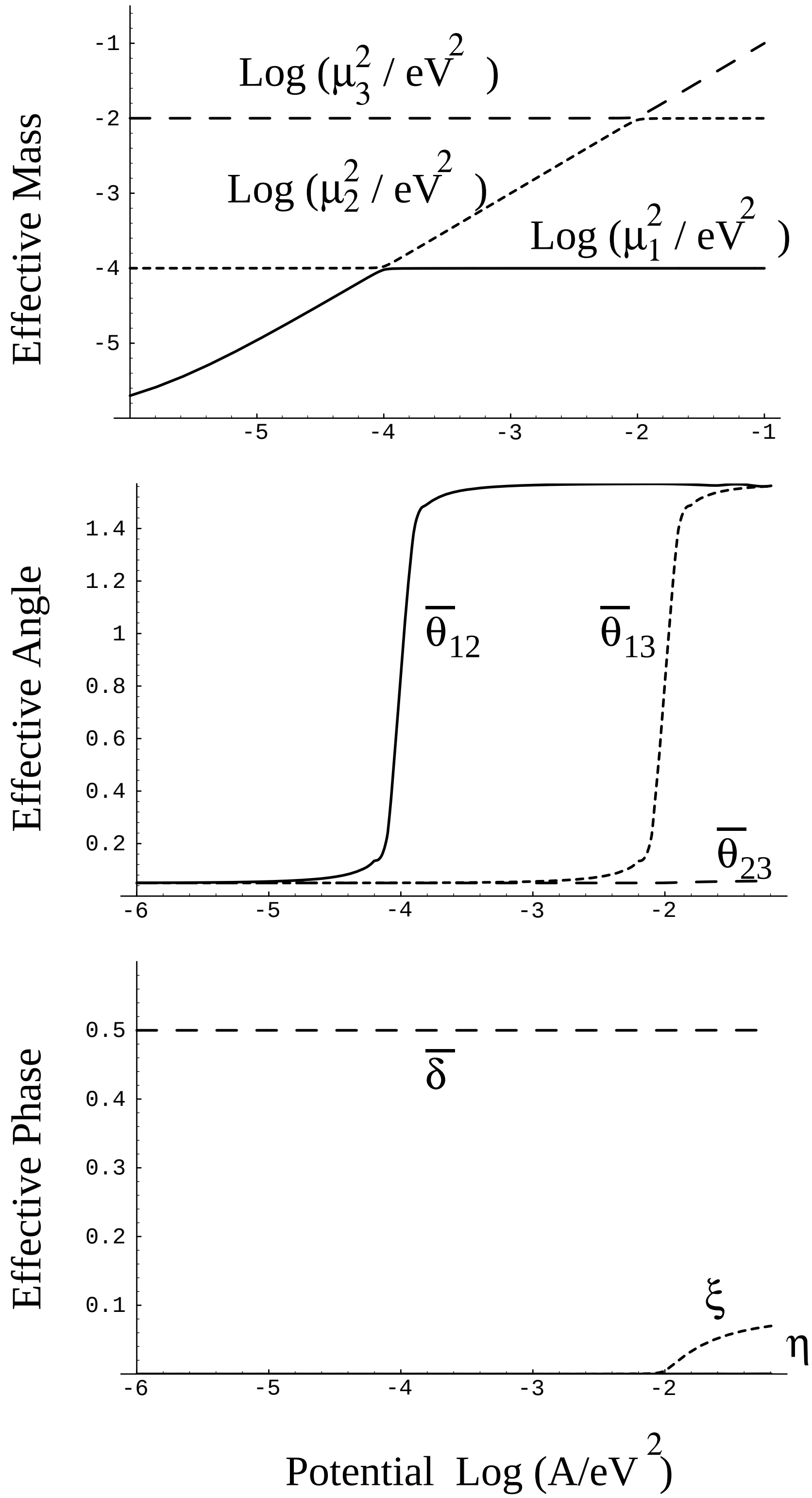
Figure Captions

Fig. 1 The effective masses, mixing angles, and CP phases in matter as functions of A with $m_1^2 = 10^{-6}\text{eV}^2, m_2^2 = 10^{-4}\text{eV}^2, m_3^2 = 10^{-2}\text{eV}^2$, mixing angles $\theta_{12} = \theta_{13} = \theta_{23} = 0.05$ and vacuum CP phase $\delta = 0.5$.

Fig. 2 The effective masses, mixing angles, and CP phases in matter as functions of A for the maximal mixing case: $m_1^2 = 10^{-6}\text{eV}^2, m_2^2 = 10^{-4}\text{eV}^2, m_3^2 = 10^{-2}\text{eV}^2$,vacuum mixing angle $\theta_{12} = \pi/4, \theta_{13} = 0.6155, \theta_{23} = \pi/4$, and vacuum CP phase $\delta = \pi/2$.

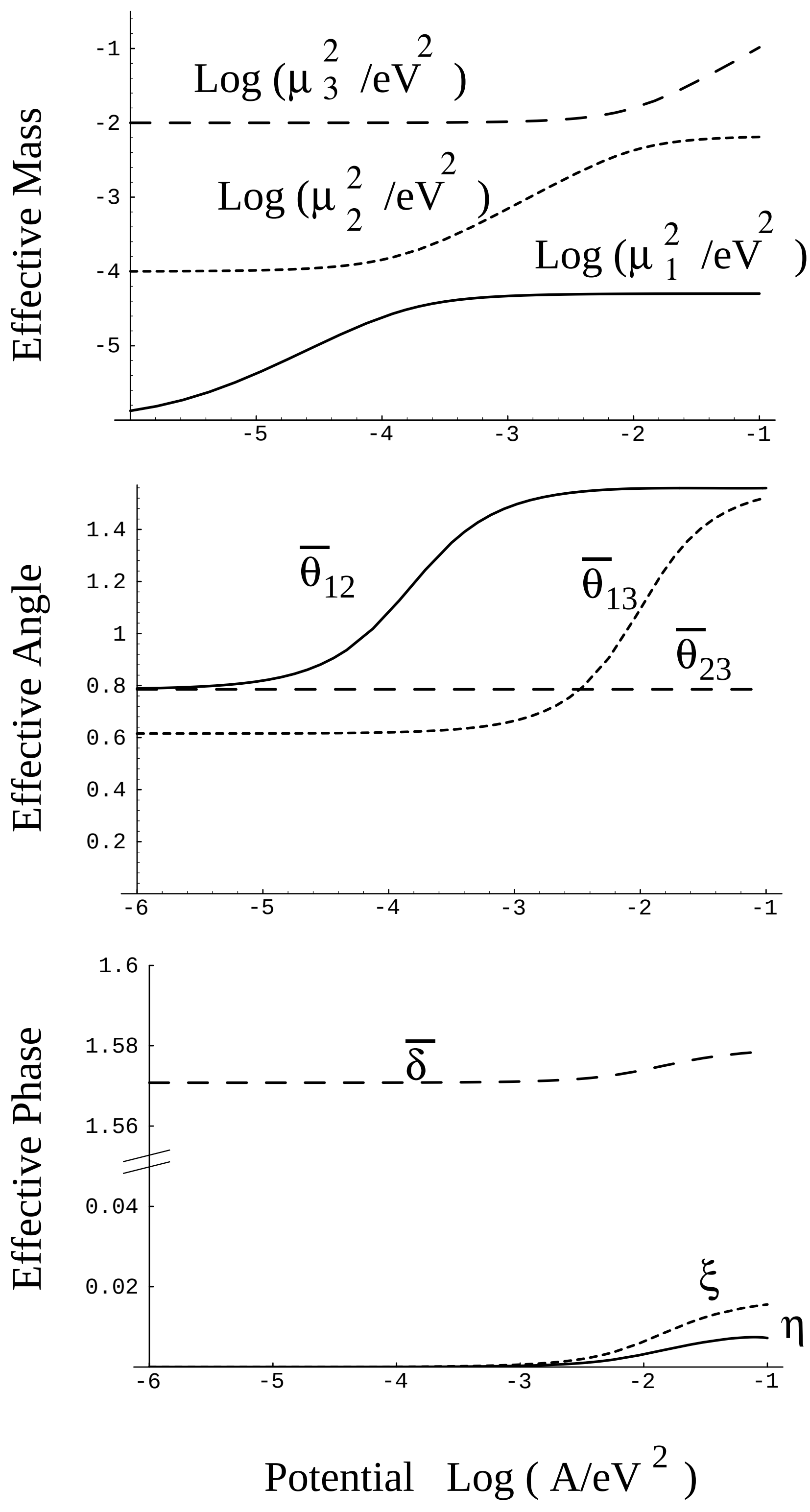
Fig. 3 The effective $(\bar{\theta})$ and dynamical (ψ) mixing angles and the dynamical mixing phase, α in the case of two generations. The parameters used are $E_\nu = 10$ MeV with (a) the vacuum mixing angle $\theta = 0.1$, $\Delta_0^2 = 10^{-4}\text{eV}^2$, (b) the vacuum mixing angle $\theta = 0.1$, $\Delta_0^2 = 10^{-6}\text{eV}^2$, and (c) the maximal mixing with two generations $\theta = \pi/2$, $\Delta_0^2 = 10^{-6}\text{eV}^2$. The abscissa x indicates the ratio of distance from the solar surface to the radius of the Sun. The value $x = 0$ means the surface of the Sun and $x = 1$ the center of the Sun. The sign of the phase α is reversed when the neutrinos travel inwards.

Fig. 4 The ν_e survival probabilities in the two generation case as functions of E_ν/Δ_0^2 for the vacuum mixing angle $\theta = 0.01$, $\theta = 0.1$, and $\theta = \pi/4$ (maximal mixing). The solid curves are the results with the use of the dynamical mixing angles whereas the dotted curves ($\theta = 0.01$, $\theta = 0.1$) are those with the use of the standard two generation formulas with the Landau-Zener transition probabilities. The Landau-Zener formula cannot be applied to the case of large mixing, specifically to the maximal mixing mixing case.



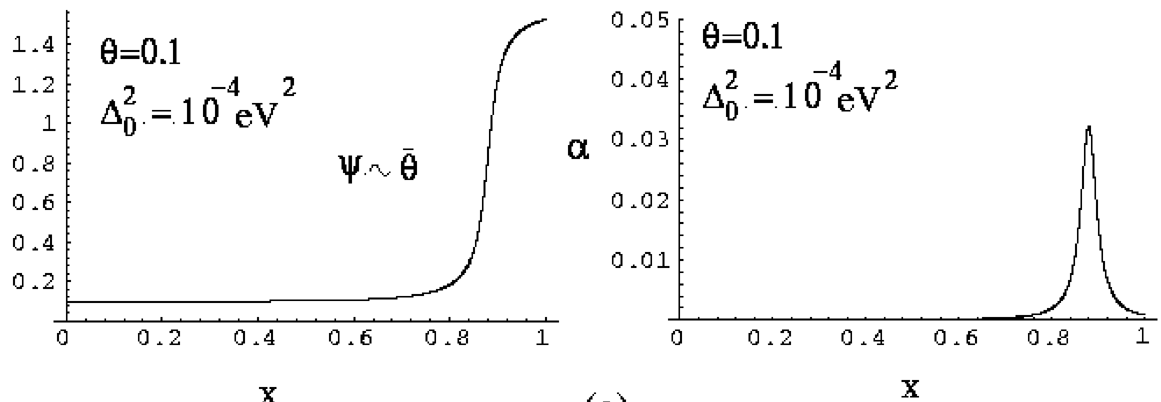
Effective Mass
Log (μ_3^2 / eV^2)
Log (μ_2^2 / eV^2)
Log (μ_1^2 / eV^2)
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
Effective Angle
1.4
1.2
1
0.8
0.6
0.4
0.2
$\overline{\theta}_{12}$
$\overline{\theta}_{13}$
$\overline{\theta}_{23}$
-6
-5
-4
-3
-2
Effective Phase
0.5
0.4
0.3
0.2
0.1
$\overline{\delta}$
ξ
η
-6
-5
-4
-3
-2
Potential Log (A/eV^2)

FIG.1

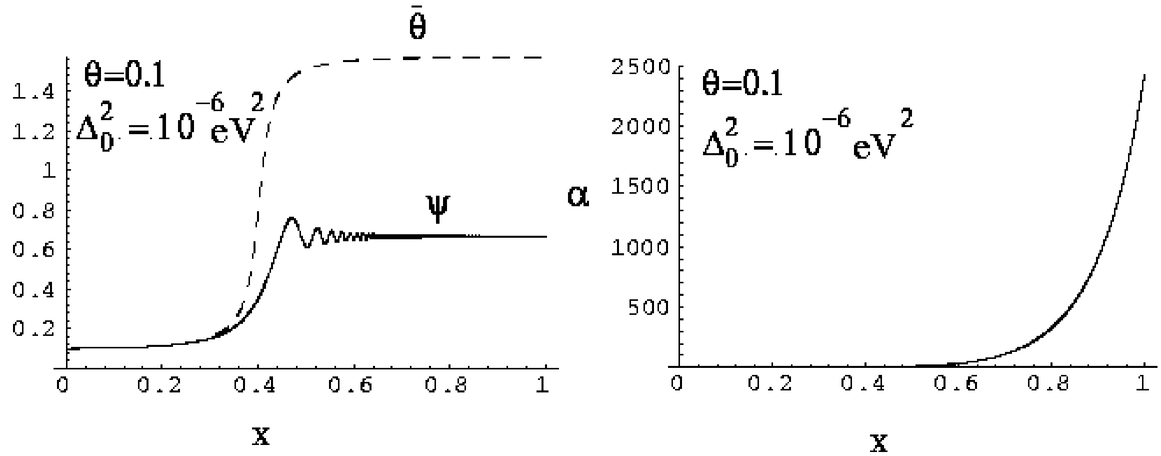


Effective Mass
Log (μ_3^2 /eV^2)
Log (μ_2^2 /eV^2)
Log (μ_1^2 /eV^2)
-1
-2
-3
-4
-5
Effective Angle
$\bar{\theta}_{12}$
$\bar{\theta}_{13}$
$\bar{\theta}_{23}$
1.4
1.2
1
0.8
0.6
0.4
0.2
Effective Phase
$\bar{\delta}$
ξ
η
1.6
1.58
1.56
0.04
0.02
-6
-5
-4
-3
-2
-1
Potential Log (A/eV ^{2})

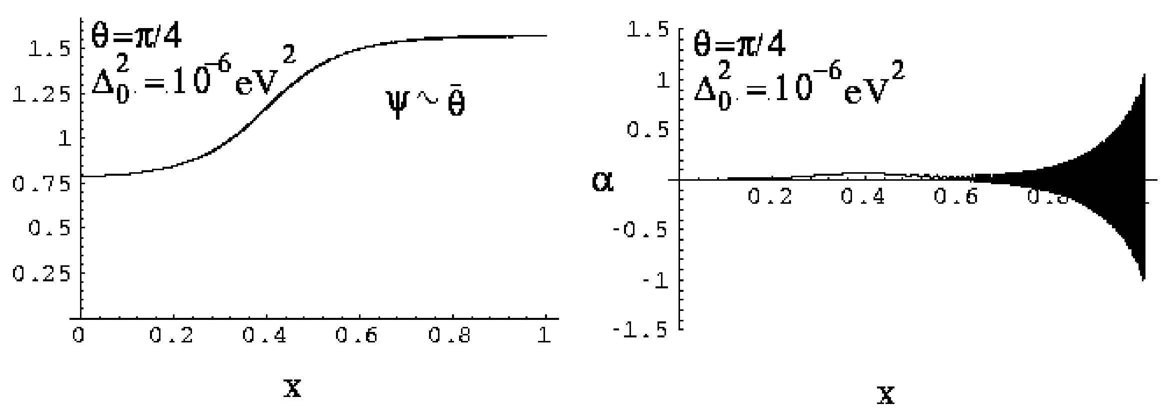
FIG.2



(a)

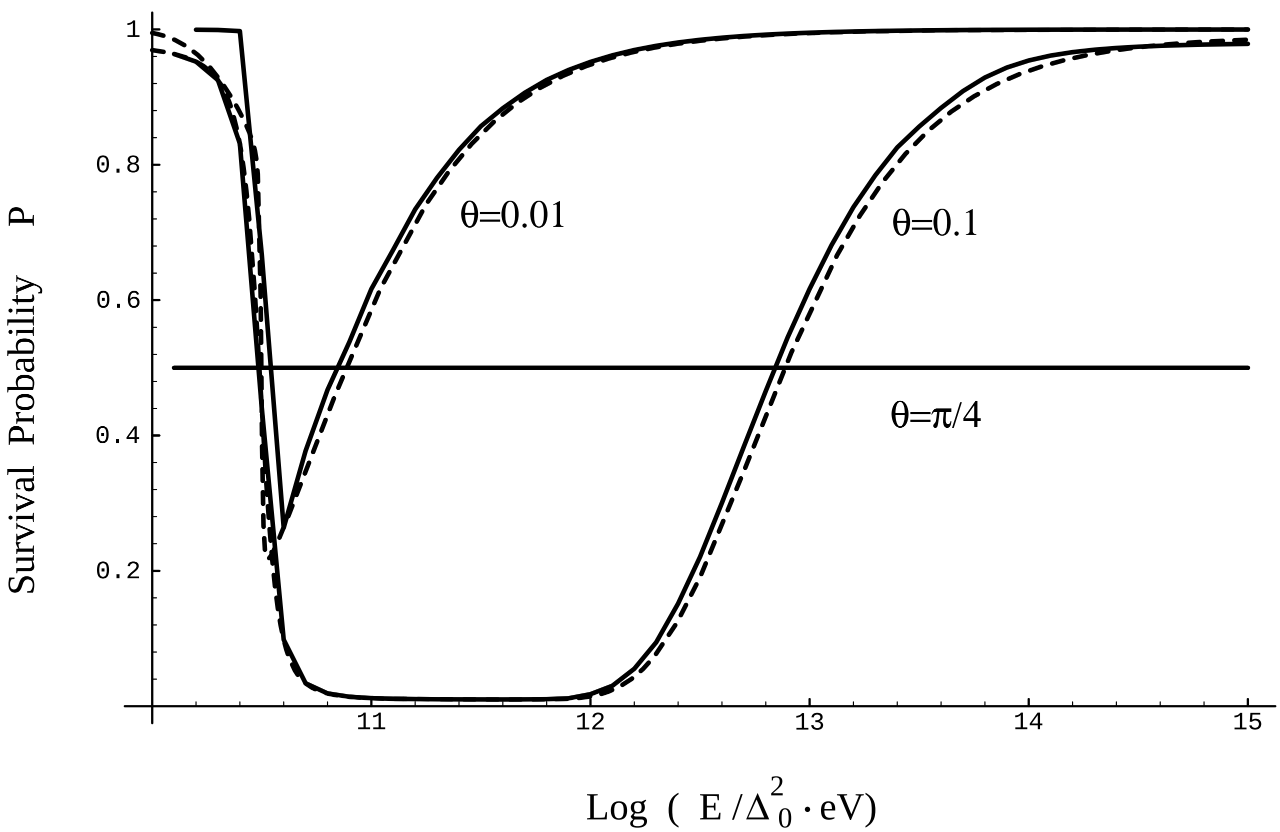


(b)



(c)

FIG.3



1
0.8
0.6
0.4
0.2
11
12
13
14
15
Survival Probability P
Log ($E/\Delta_0^2 \cdot eV$)
θ=0.01
θ=0.1
θ=π/4

FIG.4